Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Markov Chains (B3 & M2)

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Max. points : 10. **Time Limit : 1.5 hours.**

Give complete proofs. Please cite/quote appropriate results from class. You are also allowed to use results from other problems in the question paper.

Attempt any two questions only. All questions carry equal points.

Unless mentioned all state spaces are finite.

1. *Moran model with selection :* Let there be N individuals of two types - say Type I and Type II. Type I individuals have fitness level $\phi \in [1, \infty)$ and Type II individuals have fitness level 1. The population evolves as follows :

   - Given the population in generation n, an individual is chosen at random with probability proportional to its fitness level. The individual gives birth to an offspring of same type. The offspring replaces a randomly (i.e., uniformly at random) chosen individual from the exisiting population (i.e., the population at generation n), so that the total population remains constant.

   Let X_n be the number of Type I individuals in generation n. Show that $(X_n)_{n=0}^{\infty}$ is a Markov chain and find its transition matrix. Is the Markov chain irreducible and does it have a unique stationary distribution ?

2. An HMC on Ω is transitive if $\forall x, y \in \Omega$ there exists a bijection $\phi := \phi_{(x,y)} : \Omega \to \Omega$ such that

$$\phi(x) = y, \ P(z, w) = P(\phi(z), \phi(w)), \ \forall z, w \in \Omega.$$

   Show the following.

(a) The uniform distribution is a stationary distribution for transitive HMC. (**2**)

(b) The random walk on a finite group (with any increment distribution) is transitive. (**3**)

3. Let P be a transition matrix on Ω with stationary distribution π. Recall the distances $d(t) := \sup_{\mu} \|\mu P^t - \pi\|$ and $\bar{d}(t) := \sup_{\mu,\nu} \|\mu P^t - \nu P^t\|$ for $t \geq 0$ where μ, ν are probability distributions on Ω. Are $d(t), \bar{d}(t)$ decreasing in t ? If yes, prove it or else give a counter-example.